**Topology Midsemestral Exam**
**IInd semester 2017**
**B.Math. (Hons.) IInd year**
**Instructor : B.Sury**
**Maximum marks - 60**

**Q 1.** (3+4+5)
(i) Define the the left limit topology on $\mathbf{R}$ and prove that it is strictly finer than the Euclidean topology.
(ii) Show that $\mathbf{R}_l$ is first countable and separable.
(iii) Prove that $\mathbf{R}_l$ is not second countable.

**OR**

**Q 1.** (2+3+3+4)
(i) Show that the only continuous functions from ($\mathbf{R}$, Euclidean) to $\mathbf{R}_l$ are the constant functions.
(ii) Prove that the sequence $\{1/n\}$ of real numbers does not converge in the $K$-topology.
(iii) Prove that the sequence $\{-1/n\}$ does not converge in $\mathbf{R}_l$.
(iv) In the dictionary order topology of $[0,1] \times [0,1]$, determine the closure of the subset $\{(1-1/n) \times 1/2 : n \in \mathbf{N}\}$.

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**Q 2.**(3+4+5)
(i) If $C$ is a closed subset and $U$ is an open subset of a topological space $X$, prove that $C \cap U$ is open in its closure.
(ii) If $S$ is any subset of a topological space $X$, prove that the interior of $S$ is the complement of the closure of the complement of $S$.
(iii) Determine the subspace topology of a non-vertical line contained in $\mathbf{R}_u \times \mathbf{R}_u$ under the product of upper limit topologies.

**OR**

**Q 2.** (3+5+4)
Recall that a topological space $X$ is $T_1$ if, for every pair of points $x \neq y$ in $X$, there is a neighbourhood of $x$ not containing $y$ and a neighbourhood of $y$ not containing $x$.
(i) Prove that $X$ is a $T_1$-space if and only if, all singletons are closed.

(ii) Exhibit a metric space that is $T_1$ but not second countable.
(iii) Prove that every metric space is first countable.

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**Q 3.** (3+4+5)
(i) Prove that any subspace of a separable metric space is separable.
(ii) Give an example of a subspace of a separable topological space that is not separable.
(iii) Prove that in a second countable topological space, every basis contains a countable subclass which is also a basis.

**OR**

**Q 3.** (3+5+4)
(i) On the set $\mathbf{R}^{\mathbf{N}}$ of sequences, define the uniform topology.
(ii) For $n \in \mathbf{N}$, let $f_n : \mathbf{R} \to \mathbf{R}$ be continuous maps.
If $f : \mathbf{R} \to \mathbf{R}^{\mathbf{N}}; t \mapsto (f_n(t))_n$, prove that $f$ is continuous where $\mathbf{R}^{\mathbf{N}}$ has the product topology.
(iii) Give an example where (ii) fails when $\mathbf{R}^{\mathbf{N}}$ is considered with the box topology.

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**Q 4.** (3+4+5)
(i) Define the quotient topology and give an example of a quotient map that is not an open map.
(ii) Give an example of a quotient space of a Hausdorff space that is not Hausdorff.
(iii) Prove that the the cone of the $(n-1)$-sphere $S^{n-1} = \{v \in \mathbf{R}^n : ||v|| = 1\}$ is homeomorphic to the $n$-disc $D^n = \{v \in \mathbf{R}^n : ||v|| \leq 1\}$.

**OR**

**Q 4.**(4+3+5)
(i) Let $f : X \to Y$ be a continuous map into a Hausdorff space $Y$. Prove that the graph of $f$ $(\{(x, f(x)) : x \in X\})$ is a closed subset of $X \times Y$.
(ii) If $q : X \to Y$ is a quotient map that is 1-1, then show $q$ must be a homeomorphism.
(iii) Partition $\mathbf{R}^2$ into a union of concentric circles centred at $(0, 0)$; write $\mathbf{R}^2 = \cup_{r \geq 0} C_r$. Prove that $\mathbf{R}^2 \to \mathbf{R}^{\geq \mathbf{0}}; C_r \mapsto r$ is a quotient map and the quotient space is homeomorphic to $[0, \infty)$.

**Q 5.** (4+4+4)
(i) Show that if $Y$ is a connected subspace of a space $X$ and $Y \subset Z \subset \overline{Y}$, then $Z$ is connected.
(ii) Prove that $I \times I$ under the dictionary order is connected but not path-connected.
(iii) Let $f \in \mathbf{C}[X_1, \cdots, X_n]$. Prove that the set

$$\{z \in \mathbf{C}^n : f(z) \neq 0\}$$

is path-connected.

**OR**

**Q 5.**(7+5)
(i) Let $Y = \{(x, \sin(1/x) : 0 < x < 1\}$ and, let $Z$ be a arc joining $(0,0)$ to $(1, \sin(1))$ that does not intersect $Y$. Prove that $X := Y \cup Z \subseteq \mathbf{R}^2$ is path-connected but not locally connected.
(ii) Prove that $\mathbf{R}$ under the $K$-topology is connected, but not path-connected.